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Relaxation of an Unstable System Driven by a Colored Noise

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The scaling solution of an unstable system driven by a Ornstein–Uhlenbeck noise is derived from the two-variable Fokker–Planck equation. A quasiprobability distribution for the joint process (system parameter and noise) is then introduced and the system-size expansion is shown to yield a description of the relaxation process in the entire time domain.

KEY WORDS: Relaxation; unstable dynamics; colored noise; Fokker–Planck equation; scaling theory; quasiprobability distribution; system-size expansion.

1. INTRODUCTION

The relaxation process of nonequilibrium system of moderate size is conveniently modeled using stochastic differential equations (SDE). The noise terms in the SDE represent the influence of the rapidly varying degrees of freedom (almost infinite in number) on the system's "slowly" varying macroscopic parameter. Generally, the noise terms can be modeled as Gaussian white noises[(1)]; however, the need to consider the noise as an Ornstein–Uhlenbeck (OU) process in certain stochastic models for dye lasers has been demonstrated recently.[(2)] Further, stochastic evolution driven by an OU process has been of considerable theoretical interest.[(3–6)]

In this paper, we are concerned with the relaxation of a system parameter driven by a noise from an initial unstable state to the final equilibrium states via nonlinear effects.[(7–9)] In this process, usually known as unstable dynamics, one can imagine the time axis to be divided into three regimes, an initial regime where the noise initiates the relaxation process; a second regime where the nonlinear effects drive the system

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toward the final states; and a third regime where the system relaxes to the equilibrium states. The development of approximate theories of relaxation processes usually exploits the weak interaction of the noise on the system parameter and the simplest of such approaches is the system-size expansion.[1] The system-size expansion turns out to be inadequate in the treatment of unstable dynamics after the lapse of the initial regime, due to an unbounded increase in the variance of the distribution.[8,10] Therefore, there has been a great deal of interest in formulating appropriate techniques to describe (at least qualitatively) this nonlinear relaxation process.

Scaling theory[8-10] is one such approach and has successfully eplained the formation of final equilibrium states. An earlier version[8,9] completely neglected the effect of noise on the system parameter in the second and third regimes; however, this has since been remedied.[10,11] The techniques of scaling theory do not require specification of the properties of noise, such as its distribution, spectrum, etc. However, this fact has been exploited[12] in treating the unstable dynamics driven by an OU process.

An alternative approach to treat unstable dynamics in the first and second time regimes has been developed[13,14] with the introduction of a quasiprobability distribution. Furthermore, it has been shown[15] that the use of the quasiprobability distribution and system-size expansion together lead in a systematic way to the description of unstable dynamics in all the time regimes. These important developments have been given for white noise-induced relaxation and our main aim in this paper is to generalize the same to the case of an OU noise.

Our starting point is the two-variable Fokker–Planck equation (FPE) describing the evolution of the system parameter X and the OU noise U. After stating the problem in mathematical terms in Section 2, in Section 3 we derive the scaling solution valid in the first and second time regimes. The results are same as those obtained[12] based on SDE; however, we include this section since the FPE approach gives the probability distribution in a direct way. In Section 4, we introduce a quasiprobability distribution for the joint process (X, U) and show that the system-size expansion leads to a complete description of the dynamics. Thus, we generalize the approach of using a quasiprobability distribution to the case of colored noise. Finally, a brief summary is given in Section 5.

2. STATEMENT OF THE PROBLEM

We consider the relaxation of a system modeled by the SDE

$$dX/dt = h(X) + U(t) \tag{2.1}$$

where $h(x)$ is a given function (nonlinear) of X and $U(t)$ is the driving noise term. We assume that $U(t)$ can be modeled as a stationary OU process defined by the SDE

$$dU/dt = -\gamma_0 U + \eta \tag{2.2}$$

where η is a Gaussian white noise with autocorrelation function (ACF) $2\varepsilon\gamma_0^2\delta(t-t')$. The noise $U(t)$ has ACF

$$\langle U(t)\, U(t')\rangle = \varepsilon\gamma_0 \exp(-\gamma_0 |t-t'|) \tag{2.3}$$

Here ε is the noise intensity and γ_0^{-1} is the correlation time of the noise. The specific form in Eq. (2.3) is chosen so that in the limit $\gamma_0 \to \infty$, $U(t)$ reduces to a Gaussian white noise with ACF $2\varepsilon\delta(t-t')$.

Since the noise is assumed to be stationary, its probability distribution is given by

$$P_U^S = (2\pi\varepsilon\gamma_0)^{-1/2} \exp(-u^2/2\varepsilon\gamma_0) \tag{2.4}$$

We assume that the system begins to relax at $t=0$ from $x=0$, which is an unstable point, that is, $h'(0)=\gamma>0$. We use the prime to indicate the derivative with respect to the argument. Generally it may be assumed that X and U are statistically independent at $t=0$, so that their joint distribution can be written as

$$P_{\text{ini}}(x, u) = P_{\text{ini}}(x) P_U^S \tag{2.5}$$

where $P_{\text{ini}}(x)$ is Gaussian distribution with variance σ_0,

$$P_{\text{ini}}(x) = (2\pi\sigma_0)^{-1/2} \exp(-x^2/2\sigma_0) \tag{2.6}$$

The pair stochastic process (X, U) is described via the two-variable FPE[3-6]

$$\partial P/\partial t = -\partial/\partial x\{[h(x)+u]P\} + \gamma_0\, \partial/\partial u(uP + \varepsilon\gamma_0\, \partial P/\partial u) \tag{2.7}$$

We are interested in determining the conditional probability $P_c(x, u, t; x_0, u_0)$ satisfying Eq. (2.7) so that $P(x, u, t)$ can be obtained as

$$P(x, u, t) = \int P_c(x, u, t; x_0, u_0)\, P_{\text{ini}}(x_0, u_0)\, dx_0\, du_0 \tag{2.8}$$

Our aim is to determine the marginal distribution $\bar{P}(x, t)$

$$\bar{P}(x, t) = \int P(x, u, t)\, du = \int \bar{P}_c(x, t; x_0, u_0)\, P_{\text{ini}}(x_0, u_0)\, dx_0\, du_0 \tag{2.9}$$

where $\bar{P}_c$ is the integral of P_c over u. Throughout this paper we use a bar to indicate the average over the noise variable u.

3. SCALING SOLUTION FROM THE FPE

Scaling theory is an asymptotic procedure (in the limit of large t and small ε) to extract the time-dependent evolution of the probability distribution of the stochastic variable X.[10] The theory shows that the probability distribution and hence all the moments of the stochastic variable depend only on a scaled time variable τ_{sc} [see Eq. (3.15) below] in the second time regime mentioned in the introduction. The probability distribution obtained from this procedure allows one to compute the onset time, that is, the time elapsed for the onset of macroscopic order. Furthermore, the theory shows the enhancement of fluctuation in the initial and second time regimes. Different approaches[8-10] using either the Langevin equation or the FPE have been developed to derive the scaling solution. We start with the FPE and the nonlinear transformation[9,10]

$$\xi = F^{-1}[e^{-\gamma t} F(x)], \qquad \gamma = h'(0) > 0 \tag{3.1}$$

which is the solution of the deterministic equation $dx/dt = h(x)$ with the initial condition $x(0) = \xi$. The function $F(x)$ is defined by[10]

$$F(x) = \exp\left\{\int_a^x [\gamma/h(y)]\, dy\right\} \tag{3.2}$$

where a is chosen such that $F'(0) = 1$. We next transform the FPE (2.7) to the new variable ξ to obtain

$$\partial\tilde{P}/\partial t = -u\, \partial/\partial\xi[h(\xi)/h(x)\tilde{P}] + \gamma_0\, \partial/\partial u(u\tilde{P} + \varepsilon\gamma_0\, \partial\tilde{P}/\partial u) \tag{3.3}$$

where

$$\tilde{P}(\xi, u, t)\, d\xi = P(x, u, t)\, dx \tag{3.4}$$

Since $\xi = x$ at $t = 0$, the initial condition on $\tilde{P}_c$ is $\delta(\xi - \xi_0)\, \delta(u - u_0)$. The transformed FPE [Eq. (3.3)] is exact; however, it cannot be solved analytically when $h(x)$ is an arbitrary nonlinear function. We therefore introduce the scaling approximation[10]

$$h(\xi)/h(x) \approx \exp(-\gamma t) \tag{3.5}$$

which provides the asymptotic evaluation in the sense of large t and small ε mentioned earlier. Starting from the Langevin equation, it has been proved (see Section V of Ref. 10) that an approximation equivalent to that of Eq. (3.5) yields the dominant contribution to the moments of the stochastic variable in the initial and the second time regimes. With this approximation, Eq. (3.3) becomes

$$\partial\tilde{P}/\partial t = -u\, \partial/\partial\xi(e^{-\gamma t}\tilde{P}) + \gamma_0\, \partial/\partial u(u\tilde{P} + \varepsilon\gamma_0\, \partial\tilde{P}/\partial u) \tag{3.6}$$

Equation (3.6) is a two-variable FPE with linear coefficients and the solution can be obtained by standard techniques.[1] We thus get

$$\bar{\tilde{P}}_c(\xi, t; \xi_0, u_0) = (2\pi g)^{-1/2} \exp[-(\xi - f)^2/2g] \tag{3.7}$$

where

$$f(t) = \xi_0 + u_0(\gamma + \gamma_0)^{-1}\{1 - \exp[-(\gamma + \gamma_0)t]\} \tag{3.8}$$

and

$$\begin{aligned} g(t) &= 4\varepsilon\gamma_0^2[(\gamma + \gamma_0)(\gamma^2 - \gamma_0^2)]^{-1}\{1 - \exp[-(\gamma + \gamma_0)t]\} \\ &\quad - \varepsilon\gamma_0[\gamma(\gamma - \gamma_0)]^{-1}[1 - \exp(-2\gamma t)] \\ &\quad + \varepsilon\gamma_0[(\gamma + \gamma_0)^2]^{-1}\{1 - \exp[-2(\gamma + \gamma_0)t]\}, \qquad \gamma \neq \gamma_0 \\ &= \varepsilon(4\gamma_0)^{-1}[1 - \exp(-4\gamma_0 t)] - \varepsilon t \exp(-2\gamma_0 t), \qquad \gamma = \gamma_0 \end{aligned} \tag{3.9}$$

Therefore, using $\bar{\tilde{P}}_c$ of Eq. (3.7) and $\tilde{P}^{\text{ini}}(\xi, u) = P_{\text{ini}}(x, u)$ [Eqs. (2.5) and (2.6)], we obtain

$$\bar{\tilde{P}}(\xi, t) = \exp(\gamma t)(2\pi\tau)^{-1/2} \exp\{-[(\exp \gamma t)\xi]^2/2\tau\} \tag{3.10}$$

$$\tau = \exp(2\gamma t)(\sigma_0 + \sigma_\xi) \tag{3.11}$$

where

$$\begin{aligned} \sigma_\xi &= \varepsilon\gamma_0[\gamma(\gamma + \gamma_0)]^{-1} - 2\varepsilon\gamma_0(\gamma^2 - \gamma_0^2)^{-1} \exp[-(\gamma + \gamma_0)t] \\ &\quad + \varepsilon\gamma_0[\gamma(\gamma - \gamma_0)]^{-1} \exp(-2\gamma t), \qquad \gamma \neq \gamma_0 \\ &= \varepsilon(2\gamma_0)^{-1}[1 - \exp(-2\gamma_0 t)] - \varepsilon t \exp(-2\gamma_0 t), \qquad \gamma = \gamma_0 \end{aligned} \tag{3.12}$$

Next, using Eq. (3.4), it follows that

$$\begin{aligned} P(x, t) &= (2\pi\tau)^{-1/2} F'(x)[F'(F^{-1}(e^{-\gamma t}F(x)))]^{-1} \\ &\quad \times \exp\{-[e^{\gamma t}F^{-1}(e^{-\gamma t}F(x))]^2/2\tau\} \end{aligned} \tag{3.13}$$

In the scaling regime, Eq. (3.13) assumes the following form[10]:

$$\bar{P}_{\text{sc}}(x, t) \approx (2\pi\tau_{\text{sc}})^{-1/2} F'(x) \exp[-F^2(x)/2\tau_{\text{sc}}] \tag{3.14}$$

where

$$\tau_{\text{sc}} = e^{2\gamma t}\{\sigma_0 + \varepsilon\gamma_0[\gamma(\gamma + \gamma_0)]^{-1}\} \tag{3.15}$$

Equations (3.13)–(3.15) are identical in form to Suzuki's results[10] [Eqs. (6.18) and (6.21) of Ref. 10] apart from the modified definition of the scaled time variable τ. The white noise case emerges in the limit $\gamma_0 \to \infty$. The onset time[10] t_0 can be obtained from Eq. (3.4) with the condition that $\bar{P}''_{sc}(0, t_0) = 0$. The result is

$$t_0 = -[1/(2\gamma)] \ln\{F'''(0)[\sigma_0 + \varepsilon/\gamma - \varepsilon/(\gamma + \gamma_0)]\} \tag{3.16}$$

If $h(x)$ can be expanded as

$$h(x) \approx \gamma[x - \alpha x^2 - \beta x^3 + O(x^4)]$$

then $F'''(0) = (3\beta + 6\alpha^2)/\gamma$. We observe that the onset time increases with a finite correlation time γ_0^{-1} of the noise. Thus, the colored noise has a "pullback" effect on the initial relaxation of the system. The reason for this can be traced to the Gaussian form of P_U^S and the exponential ACF of U.

It is well known[10] that as $t \to \infty$, $\bar{P}_{sc}(x, t)$ approaches to delta peaks around the stable equilibrium points of the system. Thus, the scaling solution does not incorporate the asymptotic fluctuations. In the next section, we show that use of a quasiprobability distribution together with the system-size expansion leads to a systematic procedure for incorporating asymptotic fluctuation into the case of colored noise.

4. QUASIPROBABILITY DISTRIBUTION AND UNIFIED TREATMENT OF COLORED NOISE

It was pointed out earlier that the white noise-induced relaxation has been systematically treated in the entire time regime. The starting point of this approach[15] is the introduction of a quasiprobability distribution related to the distribution function of X. Following this approach, we define a joint quasiprobability distribution $Q(x, u, t)$

$$Q(x, u, t) = (2\pi\varepsilon/\gamma)^{-1/2} \int \exp[-\gamma(x - y)^2/2\varepsilon]\, P(y, u, t)\, dy \tag{4.1}$$

We note that Eq. (4.1) is not a generalization[14] of the quasiprobability distribution[13] to two dimensions, even though $P(x, u, t)$ satisfies a two-dimensional FPE. The important difference here is that the two-dimensional FPE [Eq. (2.7)] has a singular diffusion matrix.

The transformation (4.1) on Eq. (2.7) yields

$$\partial Q/\partial t = -\partial/\partial x\{[u + h(x + \varepsilon/\gamma\, \partial/\partial x)]Q\} + \gamma_0\, \partial/\partial u(uQ + \varepsilon\gamma_0\, \partial Q/\partial u) \tag{4.2}$$

To first order in noise intensity (ε), Eq. (4.2) can be approximated as

$$\begin{aligned}\partial Q/\partial t = &-\partial/\partial x[(u + h - \varepsilon h''/2\gamma)Q] - \varepsilon/\gamma\, \partial^2/\partial x^2(h'Q)\\ &+ \gamma_0\, \partial/\partial u(uQ + \varepsilon\gamma_0\, \partial Q/\partial u)\end{aligned} \tag{4.3}$$

Using Eqs. (2.5) and (2.6), we can write the initial condition on Q as

$$Q_{\text{ini}}(x, u) = P_{\text{U}}^{\text{S}}[2\pi(\sigma_0 + \varepsilon/\gamma)]^{-1/2} \exp[-x^2/2(\sigma_0 + \varepsilon/\gamma)] \tag{4.4}$$

We wish to compute the conditional probability $Q_c(x, u, t; x_0, u_0)$. The transformations for obtaining the system-size expansion[1] on Eq. (4.3) are

$$\zeta_1 = (x - Y)/\sqrt{\varepsilon}, \qquad \zeta_2 = (u - V)/\sqrt{\varepsilon} \tag{4.5}$$

The function Y and V satisfy the macroscopic equations[1]

$$dY/dt = h(Y) + V, \qquad Y(0) = x_0 \tag{4.6}$$

$$dV/dt = -\gamma_0 V, \qquad V(0) = u_0 \tag{4.7}$$

In the linear noise approximation[1] we obtain

$$\begin{aligned} \partial \Pi/\partial t = &-\partial/\partial\zeta_1\{[\zeta_2 + h'(Y)\zeta_1]\Pi\} - \gamma^{-1}h'(Y)\,\partial^2\Pi/\partial\zeta_1^2 \\ &+ \gamma_0\,\partial/\partial\zeta_2(\zeta_2\Pi + \varepsilon\gamma_0\,\partial\Pi/\partial\zeta_2) \end{aligned} \tag{4.8}$$

where $\Pi(\zeta_1, \zeta_2, t) = \varepsilon Q_c(x, u, t; x_0, u_0)$. The initial condition on Π is $\delta(\zeta_1)\,\delta(\zeta_2)$. Since our interest is in $\bar{P}(x, t)$, which is related to $\bar{Q}(x, t)$ by

$$\bar{Q}(x, t) = (2\pi\varepsilon/\gamma)^{-/2} \int \exp[-\gamma(x - y)^2/2\varepsilon]\,\bar{P}(y, t)\,dy \tag{4.9}$$

we need to compute only $\bar{Q}_c$ and hence $\bar{\Pi}(\zeta_1, t)$,

$$\bar{\Pi}(\zeta_1, t) = \int \Pi(\zeta_1, \zeta_2, t)\,d\zeta_2 \tag{4.10}$$

Equation (4.8) can again be solved by standard methods to obtain

$$\bar{Q}_c(x, t; x_0, u_0) = (2\pi\sigma_1)^{-1/2} \exp[-(x - Y)^2/2\sigma_1] \tag{4.11}$$

where σ_1 satisfies

$$d\sigma_1/dt = 2h'(Y)(\sigma_1 - \varepsilon/\gamma) + 2\sigma_2 \tag{4.12}$$

$$d\sigma_2/dt = [h'(Y) - \gamma_0]\sigma_2 + \varepsilon\gamma_0[1 - \exp(-2\gamma_0 t)] \tag{4.13}$$

with the condition $\sigma_i(0) = 0$ $(i = 1, 2)$. Therefore, $Q(x, t)$ can be written as

$$\bar{Q}(x, t) = \iint (2\pi\sigma_1)^{-1/2} \exp[-(x - Y)^2/2\sigma_1]\,Q_{\text{ini}}(x_0, u_0)\,dx_0\,du_0 \tag{4.14}$$

Note that $Y = Y(x_0, u_0, t)$ and $\sigma_1 = \sigma_1(x_0, u_0, t)$. It may be pointed out that as $\gamma_0 \to \infty$, $V \to 0$ and $\sigma_2 \to \varepsilon$ [see Eqs. (4.7) and (4.13)], hence Y and σ_1 become independent of u_0 and therefore we recover the reults of the white noise case[15] exactly. The moments with respect to $\bar{P}$ are linked[15] with the Hermite moments with respect to $\bar{Q}$ as

$$\langle X^n \rangle_P = (\varepsilon/2\gamma)^{n/2} \langle H_n[(\gamma/2\varepsilon)^{1/2} X] \rangle_{\bar{Q}} \tag{4.15}$$

Using this fact and Eq. (4.14), one can invert[12] Eq. (4.9) to yield

$$\bar{P}(x, t) = \iint [2\pi(\sigma_1 - \varepsilon/\gamma]^{-1/2} \exp[-(x - Y)^2/2(\sigma_1 - \varepsilon/\gamma)] \times Q_{\text{ini}}(x_0, u_0)\, dx_0\, du_0 \tag{4.16}$$

It may be noted that this inversion is valid only for $\sigma_1(x_0, u_0, t) > \varepsilon/\gamma$. The first two moments with respect to $\bar{P}$ can be written as

$$\langle x \rangle_{\bar{P}} = \iint Y(x_0, u_0)\, Q_{\text{ini}}(x_0, u_0)\, dx_0\, du_0 \tag{4.17}$$

$$\langle X^2 \rangle_{\bar{P}} = \iint [(\sigma_1 - \varepsilon/\gamma) + Y^2(x_0, u_0)]\, Q_{\text{ini}}(x_0, u_0)\, dx_0\, du_0 \tag{4.18}$$

which are generalizations of results for the white noise case.[15] We note that for $\gamma_0 t \gg 1$, the solution of the macroscopic equation (4.6) would become independent of the noise parameter V. Therefore, apart from a shift in the absolute values, the time dependence of the moments $\langle X^n \rangle_{\bar{P}}$ would be similar to that in the white noise limit. This fact has been shown from Monte Carlo simulation[12] of a model problem driven by the OU noise. Since Eqs. (4.17) and (4.18) reduce to their white noise counterparts for $\gamma_0 t \gg 1$, it is easy to see that they contain the asymptotic fluctuations.

5. SUMMARY

In this paper, we have derived the scaling solution for the unstable dynamics of a system parameter driven by an OU noise. Our treatment is based on the two-variable FPE for the process and has the merit of obtaining the distribution functions directly. Using the scaling solution, we have noted that the colored noise has a "pullback" effect on the initial relaxation of the system. It is reflected explicitly in the expression for the onset time. We have shown that the quasiprobability distribution originally defined for the white noise case can be generalized to incorporate a finite correlation time for the noise. It has been demonstrated that the usual system-size expansion leads, in a systematic way, to the description of colored noise induced dynamics in the entire time domain. It has been argued that the finite correlation time of the OU noise has a significant effect on the average properties only in the initial time regime.

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